Statistics Review

Chapter 3, with 2.5/2.6

EC200: Econometrics and Applications

Learning objectives

- Understand and use key vocabulary (Chapter 3)
- Construct confidence intervals
- Conduct one and two-sided hypothesis tests
  - Using z- and t- distributions
  - Interpret p-values

Statistics Review

Random sampling

Simple random sampling — Definition

Method of choosing a set of observations (sample) from a population, such that each member is **equally likely** to be included.
We label each of n observations as Y_1, Y_2, ... Y_n

Independent and identically distributed (i.i.d.) — Definition

When Y_1, Y_2, ... Y_n are

1. drawn from the same distribution *(identical)*, and
2. are independent (conditional = marginal distribution)

With simple random sampling, the random variables Y_i are i.i.d.

Finite sample properties of estimators

- An estimator of a population parameter is a random variable that depends on sample information, whose value approximates this parameter
- A specific value of that random variable is an estimate.

Example 1

Draw a sample of size n from a population, with parameter μ. One useful estimator:

$$\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$$

$\bar{Y}$ is an estimator, and $\bar{y}$ is the estimate. A sampling distribution is the distribution of an estimator.

Law of large numbers

Law of large numbers Definition

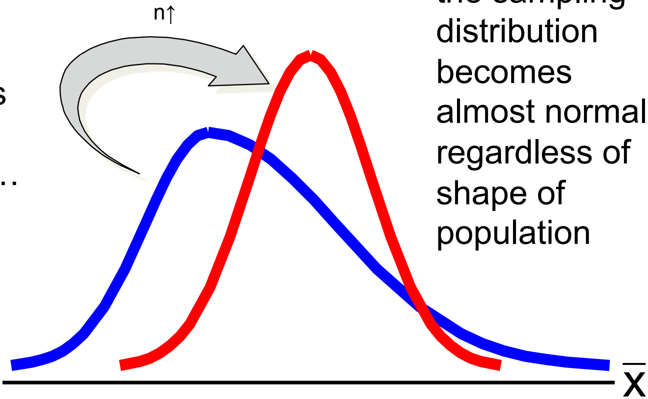
If Y_i, $i = 1, ..., n$ is i.i.d, with $E(Y_i) = \mu_Y$ and if large outliers are unlikely (if $var(Y_i) = \sigma_Y^2 < \infty$), then

$$\bar{Y} \xrightarrow{p} \mu_Y$$

That is, $\bar{Y}$ "converges in probability" to μ_Y. Alternatively, we can say that $\bar{Y}$ "is consistent" for μ_Y

Central limit theorem

As the sample size gets large enough…



the sampling distribution becomes almost normal regardless of shape of population

Central limit theorem

Central limit theorem
Definition

- Let X_1, X_2, ... X_n be a set of n independent random variables with identical distributions with mean μ and variance σ^2, and $\bar{X}$ is the mean of these random variables
- As n becomes large, the distribution of

$$Z = \frac{\bar{X} - \mu_X}{\sigma_{\bar{X}}}$$

approaches the standard normal distribution (is "asymptotically normal")

Characteristics of point estimators

We evaluate how good an estimator is based on its bias and efficiency:

- Bias: Difference between the expectation of the estimator and the parameter
- Efficiency: Variance of the estimator - how much it differs from the true parameter

Bias

Let $\hat{\theta}$ be an estimator of parameter θ:

Bias — Definition

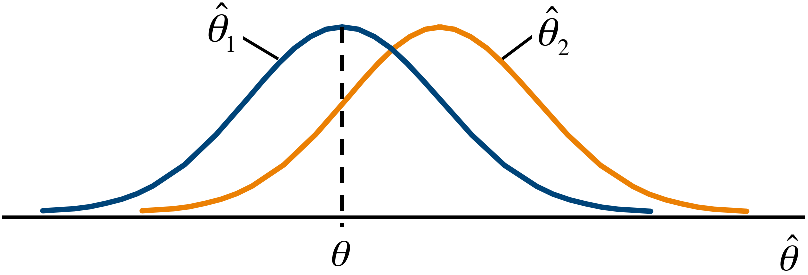
The difference between the expectation of the estimator and the parameter

$$Bias(\hat{\theta}) = E[\hat{\theta}] - \theta$$

The bias of an unbiased estimator is 0.

Unbiasedness

$\hat{\theta}_1$ is an unbiased estimator, $\hat{\theta}_2$ is biased:



Efficiency

- Often, there are several unbiased estimators.
- Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators of θ. Then, $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$ if

$$\boxed{Var(\hat{\theta}_1) < Var(\hat{\theta}_2)}$$

Confidence limits for μ

Confidence interval:

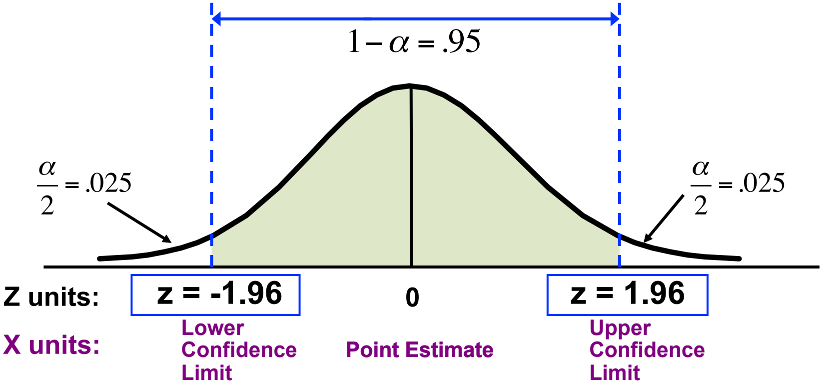
$$\boxed{\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}}$$

where $z_{\alpha/2}$ *is the normal distribution value for the probability of* $\alpha/2$ *in each tail*

If σ *unknown, then use the* t *distribution instead*

Finding $z_{\alpha/2}$

Consider a 95% confidence interval:



CI Example

Example 2

A sample of 11 circuits from a large, normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is 0.35 ohms.

Determine a 95% confidence interval for the true mean resistance of the population.

CI Example

A sample of 11 circuits from a large, normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is 0.35 ohms. Find a 95% CI for the true mean resistance of the population.

1. List what we know:

$$n = 11 \qquad \bar{x} = 2.20$$

$$\sigma = 0.35 \qquad \alpha = 0.05$$

   population normal

2. List what we want to find:

$$\bar{x} \pm z\frac{\sigma}{\sqrt{n}}$$

CI Example

A sample of 11 circuits from a large, normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is 0.35 ohms. Find a 95% CI for the true mean resistance of the population.

3. Find the right value of $z_{\alpha/2}$:
   $\alpha = 0.05 \Rightarrow z_{0.05/2} \Rightarrow P(Z < z_{0.025}) = 0.975 \Rightarrow z_{0.025} = 1.96$
4. Plug in remaining values:

$$95\%CI = 2.20 \pm 1.96\frac{0.35}{\sqrt{11}}$$
$$= 2.20 \pm 0.2068$$
$$1.9932 < \mu < 2.4068$$

Concepts of hypothesis testing

A hypothesis is a claim (assumption) about a population parameter:

- One sample: *The mean monthly cell phone bill in Vermont is* $\mu = \$52$.
- Two sample: *The mean monthly cell phone bill in Vermont equals the mean monthly cell phone bill in Massachusetts.*

Setting up hypotheses

- Null hypothesis (H_0) states the assumption (numerical) to be tested
- Alternative hypothesis (H_1) is the "opposite" of the null
- Determine whether there is enough evidence to reject the null hypothesis.
- Example: *The average number of TV sets in U.S. homes equals three* ($H_0 : \mu = 3$, $H_1 : \mu \neq 3$).

$$H_0 : \mu = 3 \qquad\qquad H_0 : \cancel{\bar{x} = 3}$$

One-tail tests

In many cases, the alternative hypothesis focuses on one particular direction.

- Does fuel additive *increase* gas mileage?

$$\boxed{\begin{array}{c} H_0 : \mu \leq 10.5 \\ H_1 : \mu > 10.5 \end{array}}$$

Upper-tail test since alternative hypothesis focused on upper tail.

- Does cholesterol drug *lower* LDL levels from average of 145?

$$\boxed{\begin{array}{c} H_0 : \mu \geq 145 \\ H_1 : \mu < 145 \end{array}}$$

Lower-tail test since alternative hypothesis focused on lower tail.

Two-tail tests

Sometimes, we don't have a specific direction in mind.

- Were average U.S. stock market returns affected by Hurricane Katrina, compared to their usual average of 4%?

$$\boxed{\begin{array}{c} H_0 : \mu = 4 \\ H_1 : \mu \neq 4 \end{array}}$$

Two-tailed test since we reject if stock returns are very high or very low

Level of significance, α

- Significance level defines the unlikely values of the sample statistic, the rejection region, if the null hypothesis is true
- Designated by α (level of significance) - usually $\alpha = 0.01, 0.05, 0.10$
- Selected by researcher at beginning
- Determines the critical value of the test

Step-by-step

1. Set up H_0 and H_1
2. Determine t-statistic:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

3. Compare test statistic to critical value(s) c, depends on α and one vs. two-sided test
   a. Upper tail: Reject H_0 if $t > c$
   b. Lower tail: Reject H_0 if $t < -c$
   c. Two tailed: Reject H_0 if $|t| > c$
4. Reject or do not reject H_0

Test statistics and critical values

We essentially "convert" our estimate to the t-distribution:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

If we know σ or as n gets large, the t distribution converges to a standard normal (z) distribution.

P-values

P-value Definition

The largest significance level at which we could carry out a hypothesis test and still fail to reject the null hypotheses.

- Also called "observed level of significance"
- Smallest value of α for which we can reject H_0

Example: Hypothesis test for mean

Example 3

A phone industry manager things that customer monthly cell phone bills have increased and now average over \$52 per month.

- The company wishes to test this claim, so it surveys 150 customers.
- The average phone bill is \$53.10 per month, with a standard deviation of \$10.
- Test the null hypothesis that bills have not increased at the 5% level.

Example: Hypothesis test for mean

1. Write down what we know:
   - $\mu_0 = 52$ $s = 10$, $n = 150$
   - $\alpha = 0.5$, $\bar{x} = 53.1$
2. Set up hypotheses:
   - H_0: $\mu \leq 52$
   - H_1: $\mu > 52 \rightarrow$ *what manager wants to prove*
   - This is an *upper* tail test

Example: Hypothesis test for mean

3. Since we have a upper-tail test, we will reject if we have a t-test statistic greater than t_α.
4. Decision rule: Reject H_0 if $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} > 1.96$
5. Reject or do not reject:

$$z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{53.1 - 52}{10/\sqrt{150}} = 1.347$$

DO NOT REJECT H_0

Calculate the p-value

3. Convert $\bar{x}$ to test statistic $\Rightarrow 1.347$
4. Calculate p-value

$$\begin{aligned} P(Z > 1.347) &= 1 - F(1.35) = 1 - 0.9115 \\ &= 0.0885 \end{aligned}$$

5. Do not reject, as $\alpha = 0.05 < 0.0885 = p$. Can reject only at significance level of 0.0885 or higher.

Difference between two means

- We may want also want to compare the means of two different population distributions.
  - Do average study hours differ between first-year and upper-year students?
  - Does a new drug lower blood pressure better than a placebo?
- All intuition is the same, with modest changes.

Difference between two means

- Now we test $H_0 : \mu_1 - \mu_2 = d_0$ vs $H_1 : \mu_1 - \mu_2 \neq d_0$
- Often $d_0 = 0$ because we want to know if there is a difference. We collect information on $\bar{X}_1$ and $\bar{X}_2$, along with s_1 and s_2

Difference between two means

- Because these are drawn from separate populations, they are independent random variables.
- CLT: $\bar{Y_1} \sim N(\mu_1, \sigma_1^2/n_1)$ and $\bar{Y_2} \sim N(\mu_2, \sigma_2^2/n_2)$
- Since independent:
  $\bar{Y_1} - \bar{Y_2} \sim N((\mu_1 - \mu_2), (\sigma_1^2/n_1) + (\sigma_2^2/n_2))$
- But, we don't know σ_1 or σ_2!

Difference between two means

For our purposes, we will use the following estimator of the standard error of the difference between these two independent random variables:

Definition 4

$$SE(\bar{Y_1} - \bar{Y_2}) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Difference between two means

Now we can calculate a t-statistic!

Definition 5

$$t = \frac{(\bar{Y_1} - \bar{Y_2}) - d_0}{SE(\bar{Y_1} - \bar{Y_2})}$$

$$t = \frac{(\bar{Y_1} - \bar{Y_2}) - d_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

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Summary

1. Finite sample properties of estimators
2. Confidence intervals
3. Hypothesis testing
   - Overview
   - P-values
4. Comparing means from different populations